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Mean field theory and geodesics in general relativity

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Abstract. It is proven that all geodesics in a mean gravitational field can be interpreted locally as the averages of geodesics in the unaveraged field. The respective time-like, space-like or null character of averaged and unaveraged geodesics is discussed carefully. Finally, some important astrophysical and cosmological applications and consequences are investigated.

PACS. 03.50.-z Classical field theories – 05.20.Dd Kinetic theory – 04.20.-q Classical general relativity

Notations and terminology

In this article, space-time indices running from 0 to 3 will be indicated by Greek letters. The metric signature will be $(+,-,-,-)$. I also have chosen, as a rule, *not* to use the so-called intrinsic notation in differential geometry and to stick to the notation standard in physics, which denotes every tensor by its components. Moreover, since each curve considered in this article will only be studied in a single parametrization, no systematic explicit distinction between curves and graphs has been deemed necessary.

1 Introduction

General relativity is commonly believed to be *the* right theory of non-quantum gravitation [16]. It models [15] the physical space-time by a 4D differentiable manifold, endowed with both a metric and a connection. Einstein's theory further assumes that the connection is completely determined by the metric and that both are linked to the stress-energy tensor of the matter present in space-time through the so-called Einstein equation. The general relativistic gravitational field is thus essentially a metric defined on the space-time manifold or, equivalently, a metric and its associated Levi-Civita connection.

Mean field theory plays an important role in practically every branch of physics; it therefore comes as no surprise that developing a mean field approach to gravitation has been the subject of active research for more than a decade (see for example [2,5,6,8,17]). This conceptually and practically crucial problem has been recently solved in quite general a way. It has been shown [3] that, given a statistical ensemble Σ of space-times sharing a common topology, it makes both mathematical and physical sense to define the mean space-time associated to this ensemble as a space-time of the same topology, but where the gravitational field is represented by a metric which is simply the average of the metrics corresponding to the various space-times members of Σ.

This mean-metric defines geodesics on the physical space-time; these are the geodesics of the mean gravitational field. In particular, those which are time-like or null can be interpreted as the trajectories of point-like particles in the mean gravitational field. It then seems only natural to wonder if there is any correspondence between these geodesics (be they time-like, null or space-like) and the geodesics in the original space-times members of the statistical ensemble Σ. In particular, is it possible to view each geodesic of the mean space-time as the statistical average of a collection of geodesics, each member of the collection being a geodesic in one of the space-times in Σ?

The aim of this article is to elucidate this question in as general a manner as possible and to discuss some connected astrophysical and cosmological problems. The matter is organized as follows. Section 2 is devoted to reviewing some basic definitions and results concerning statistical ensembles of space-times and the mean gravitational fields with which they are associated. Section 3 concerns the very definition of geodesics; part of the material presented there is naturally not new, but it has been included to make this article as self-contained as possible. Let us be more precise about the contents of Section 3. On a differentiable manifold where connection and metric are two a priori independent fields, one can construct two generally different sets of curves which might be both considered as geodesics. One set depends on the metric and the other one is entirely fixed by the connection. In a general relativistic space-time, where the connection is the Levi-Civita connection of the metric, both sets coincide and one thus has two completely equivalent but computationally very different definitions of geodesics. Section 3 elaborates on this point and makes clear which of these

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two definitions is best suited for studying the workings of the averaging procedure described in Section 2.

The main results of this article are presented in Section 4. Let Σ be a statistical ensemble of space-times with common topology. Given an arbitrary point M and an arbitrary vector U at this point, one can consider, for each space-time in Σ, the unique geodesic which passes through M with tangent U at M. By varying the space-time in Σ, one obtains a collection $\sigma_{M,U}$ of geodesics which depends on both M and U. To the best of my knowledge, the existing literature does not provide with a general, geometrically natural method to average, at least locally, such a collection of curves into an intrinsically[1] defined 'average curve'. It is argued in Section 4 that this problem is solved by considering the geodesics of the mean metric.

Indeed, given an arbitrary chart $\mathcal{C}$ common to all space-times in Σ around point M (see Sect. 2.1), it is possible to define at least two generally different curves of the mean space-time which one might be tempted to view as averages of the geodesics in the collection $\sigma_{M,U}$. Loosely speaking, the first of these curves, $\chi_{\mathcal{C}}$, is obtained by averaging the coordinates of the points on the geodesics while the second one, $\xi_{\mathcal{C}}$ is obtained by averaging the momentum components. The retained notations[2] emphasize that these two 'average curves' generally depend on the choice of the chart $\mathcal{C}$. In other words, these curves are not intrinsic objects and different coordinate choices will generally lead to different average curves.

The principal result of Section 4 can now be stated as follows. Given an arbitrary neighborhood $\mathcal{N}$ of point M, the unique geodesic $\bar{\gamma}_{M,U}$ of the mean space-time which passes through M with 'velocity' U provides, for all charts $\mathcal{C}$ defined on $\mathcal{N}$, a respectively first order and second order approximation to the curves $\chi_{\mathcal{C}}$ and $\xi_{\mathcal{C}}$. Since $\bar{\gamma}_{M,U}$ does not depend on the choice of a chart, this geodesic can be viewed as *the* natural, intrinsically defined curve averaging the geodesics of the collection $\sigma_{M,U}$ in the neighborhood $\mathcal{N}$ of M. Because the result is moreover valid for arbitrary M and U, this also proves that any (given) geodesic of the mean space-time can be viewed locally as the statistical average of a collection of geodesics belonging to the various members of Σ. Appendix A elaborates a little further on the definitions of $\chi_{\mathcal{C}}$ and $\xi_{\mathcal{C}}$

The result of Section 4, together with some related astrophysical and cosmological problems, is discussed in Section 5. Section 5.1 elaborates on the respective time-like, space-like or null character of the various geodesics involved in the averaging. Section 5.2 provides some rough calculations and orders of magnitude that strongly suggest that using a mean field theory to study geodetic motions of particles on a cosmological scale does furnish fairly accurate results. Section 5.3 discusses how the Sachs equations model the evolution of a beam of photons and justifies the use of the mean field approximation in these equations. Finally, Section 6 provides a summing up of the contents of this article.

[1] The definitions retained in this article for the terms 'intrinsic' and 'covariant' are given in Appendix A.

[2] The notations $\chi_{M,U,\mathcal{C}}$ and $\xi_{M,U,\mathcal{C}}$ would certainly be more accurate than the chosen ones but they have been judged unnecessarily cumbersome since all the geodesics being averaged pass through a common point M with the same 'velocity' U.

2 Mean gravitational field

2.1 Ensembles of space-times

Let us consider a statistical ensemble Σ of space-times $\mathcal{M}(\omega)$, $\omega \in \Omega$; Ω is a measured but otherwise arbitrary set (probability set [7]). Each member of the ensemble Σ is a differentiable manifold endowed with a metric $g(\omega)$ and a connection $\Gamma(\omega)$. The covariant derivative operator defined by the connection coefficients $\Gamma(\omega)$ will be denoted $\nabla(\Gamma(\omega))$.

We will restrict the discussion by supposing that, given any two space-times $\mathcal{M}(\omega_1)$ and $\mathcal{M}(\omega_2)$ in Σ, there always exists a one-to-one bi-continuous mapping between their points[3]. One can then choose an atlas common to all manifolds in Σ and, for any chart, (i.e. any local coordinate system (x)), represent every member of the statistical ensemble Σ by an ω-dependent metric field $g(x,\omega)$ and ω-dependent connection coefficients $\Gamma^{\alpha}_{\mu\nu}(x,\omega)$. In a somewhat less precise, but more physically oriented language: all space-times in the ensemble have a single, common topology and are distinguished only by their respective gravitational fields.

Each space-time $\mathcal{M}(\omega)$ is an Einstein space-time. This means that its metric and connections verify the equations of general relativity. The first of these equations [15] expresses the so-called compatibility of the metric with the connection; one has, for each $\omega \in \Omega$:

$$\nabla_\mu(\Gamma(\omega))\ g_{\alpha\beta}(\omega) = 0 \tag{1}$$

for any $\mu, \alpha, \beta = 0, 1, 2, 3$. This implies that the $\Gamma(\omega)$'s can be expressed in terms of the metric tensor $g(\omega)$ and of its partial derivatives [15]:

$$\Gamma^{\alpha}_{\mu\nu}(\omega) = \frac{1}{2}\ g^{\alpha\beta}(\omega)\left(\frac{\partial g_{\beta\nu}(\omega)}{\partial x^\mu} + \frac{\partial g_{\beta\mu}(\omega)}{\partial x^\nu} - \frac{\partial g_{\mu\nu}(\omega)}{\partial x^\beta}\right). \tag{2}$$

In (2), the $g^{\alpha\beta}(\omega)$'s are the coordinate basis components of the tensor $g^{-1}(\omega)$ inverse to the metric $g(\omega)$.

The connection can also be represented by another set of coefficients [10]. One defines:

$$\Gamma_{\mu,\alpha\beta}(\omega) = g_{\mu\nu}(\omega)\Gamma^{\nu}_{\alpha\beta}(\omega) \tag{3}$$

and one finds immediately from (2) that:

$$\Gamma_{\mu,\alpha\beta}(\omega) = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}(\omega)}{\partial x^\beta} + \frac{\partial g_{\mu\beta}(\omega)}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}(\omega)}{\partial x^\mu}\right). \tag{4}$$

[3] Such a mapping is called an homeomorphism [11] and its existence traces the fact that $\mathcal{M}(\omega_1)$ and $\mathcal{M}(\omega_2)$ have the same topology.

The second equation [15] of general relativity is the Einstein equation proper:

$$\mathcal{E}_{\mu\nu}(\nabla(\Gamma(\omega)), g(\omega)) \equiv R_{\mu\nu}(\omega) - \frac{1}{2}R(\omega)g_{\mu\nu}(\omega) = \chi g_{\mu\alpha}(\omega)g_{\nu\beta}(\omega)\, T^{\alpha\beta}(\omega). \quad (5)$$

The $R_{\mu\nu}$'s are the coordinate-basis components of the Ricci-tensor, R is the trace of this tensor, T is the stress-energy tensor of the matter in space-time and χ is the gravitational constant. The combination on the left-hand side of (5) is usually called the Einstein-tensor of the metric and the connection, hence the notation.

2.2 Definition of a mean space-time

It has been shown in [3] that the statistical ensemble Σ of space-times can be used to define a single, mean Einstein space-time $\bar{\mathcal{M}}$ and that, by construction, the atlas common to all members of Σ can be used as an atlas for $\bar{\mathcal{M}}$. As all Einstein space-times, $\bar{\mathcal{M}}$ is characterized by a metric $\bar{g}$ and a connection $\bar{\Gamma}$ which obey the equations of general relativity. The metric $\bar{g}$ is the average of the metrics $g(\omega)$ over ω; one thus has, for all x:

$$\bar{g}(x) = \langle g(x, \omega)\rangle, \quad (6)$$

where the brackets on the right-hand side indicate an average over the statistical ensemble Σ. The connection $\bar{\Gamma}$ is simply the connection compatible with the metric $\bar{g}$. Since equation (4) is linear in both the metric g and the 'covariant' connection coefficients, one simply has, for all x:

$$\bar{\Gamma}_{\mu,\alpha\beta}(x) = \langle \Gamma_{\mu,\alpha\beta}(x, \omega)\rangle\,. \quad (7)$$

This naturally entails that the Christoffel symbols of the mean connection are *not* identical to the averages of the Christoffel symbols associated to the various space-times $\mathcal{M}(\omega)$ members of Σ. A thorough discussion of the mathematical and physical motivations for definition (6) can be found in [3].

Because the Einstein tensor depends non linearly on the connection and the metric, the Einstein tensor $\bar{\mathcal{E}} = \mathcal{E}(\nabla(\bar{\Gamma}), \bar{g})$ associated to the mean connection and mean metric does not generally coincide with the average of the Einstein tensors $\mathcal{E}(\nabla(\Gamma(\omega)), g(\omega))$. The tensor $\bar{\mathcal{E}}$ is nevertheless the Einstein tensor of the mean space-time. It therefore defines, via Einstein equation, a stress-energy tensor $\bar{T}$ for the mean space-time. The tensor $\bar{T}$ can naturally be expressed as the average of a certain tensor τ over the statistical ensemble Σ; one thus has, for all x, $\bar{T}(x) = \langle \tau(x, \omega)\rangle$. The exact expression for $\tau(\omega)$ in terms of $T(\omega)$, $g(\omega)$ and $\Gamma(\omega)$ is given in [3]. This expression actually describes the change in the equation of state of the matter upon averaging over the statistical ensemble of space-times. As shown in [3], this change is highly non trivial. In particular, the vanishing of $T(\omega)$ for all ω does not necessarily imply the vanishing of $\tau(\omega)$. The mean stress-energy tensor $\bar{T}$ can therefore be non vanishing in regions where the unaveraged stress-energy tensor actually vanishes. A complete general discussion of this and other perhaps unexpected consequences of definition (6) can be found in [3]. The particular cases where the matter is made of an electromagnetic field and/or of a possibly charged perfect fluid is also addressed in depth by [3].

3 Geodesics

Let us consider an arbitrary differentiable manifold $\mathcal{M}$ endowed with a connection Γ and a metric g not necessarily compatible with Γ ($\mathcal{M}$ is therefore *not* supposed to be an Einstein space-time). One can consider on $\mathcal{M}$ two generally different kinds of geodesics.

3.1 Geodesics of the connection

A first set $\mathcal{S}_\Gamma$ of geodesics is made of those curves whose tangent vector is parallel propagated along itself. This set of curves depends only on the connection Γ, and not directly on the metric g. It can be shown [15] that, after proper parametrization, the differential equations of these curves $x(\lambda)$ read:

$$u^\mu = \frac{dx^\mu}{d\lambda}, \quad (8)$$

and

$$\frac{du^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta. \quad (9)$$

The geodesics in $\mathcal{S}_\Gamma$ may be called the geodesics of the connection. Equations (8) and (9) are equally valid for space-like, time-like and null geodesics.

3.2 Geodesics of the metric

The second set of geodesics $\mathcal{S}_g$ can be defined through a variational principle which involves only the metric g and not the connection. These geodesics will be called the geodesics of the metric. One can actually construct several variational principles [1,12,15] which all deliver these same geodesics, but with possibly different parametrizations. For the purposes of this article, only one of these principles needs to be considered and I will discuss it now in full detail. This variational principle presents the great advantage of applying equally to time-like, space-like and null geodesics.

Let us fix two points M and N with coordinates X and Y in $\mathcal{M}$ and consider the curves $x(\lambda)$ which pass through M and N ($x(a) = X$, $x(b) = Y$) and extremize the 'action':

$$\mathcal{A}\left\{X, Y, x(\lambda),\ \frac{dx}{d\lambda}\right\} = \frac{1}{2}\int_a^b g_{\mu\nu}(x(\lambda))\,\frac{dx^\mu}{d\lambda}\,\frac{dx^\nu}{d\lambda}\,d\lambda = \int_a^b L\left\{x(\lambda),\ \frac{dx}{d\lambda}\right\} d\lambda. \quad (10)$$

The use of braces (as opposed to round brackets) in (10) takes into account the fact that both $\mathcal{A}$ and L are functionals of $x(\lambda)$, and not functions. The 'momentum' $p(\lambda)$ conjugated to $x(\lambda)$ is:

$$p_\mu(\lambda) = \frac{\delta L}{\delta \dot{x}^\mu} = g_{\mu\nu}(x(\lambda))\ \frac{dx^\nu}{d\lambda} \tag{11}$$

where the standard notation $\dot{x} = \dfrac{dx}{d\lambda}$ has been used. This can be inverted to give:

$$\frac{dx^\mu}{d\lambda} = g^{\mu\nu}(x(\lambda))\ p_\nu(\lambda). \tag{12}$$

The 'equation of motion' for p reads:

$$\frac{dp_\mu}{d\lambda} = \frac{\delta L}{\delta x} = \frac{1}{2}\ \partial_\mu g_{\alpha\beta|x(\lambda)} \frac{dx^\alpha}{d\lambda}\ \frac{dx^\beta}{d\lambda}. \tag{13}$$

Equations (11) and (13) apply equally to the space-like, time-like and null geodesics; so does the 'action' $\mathcal{A}$. A parametrization of the geodesic for which equation (13) applies is said to be affine.

3.3 Coincidence of the two sets of geodesics in Einstein space-times and consequence

All the space-times that will be considered in the remainder of this article will be Einstein space-times, endowed with a metric g and a connection Γ compatible with each other. It is a standard result (see for example [15]) that the two sorts of geodesics coincide in these space-times and they will be called simply the 'geodesics' of the space-time. Let us remark that a parametrization for which (9) applies is then affine, in the sense introduced of Section 3.1.

The geodesics of Einstein space-times can thus be described by two completely equivalent equations. The first equation (9) involves only the usual connection coefficients (Christoffel symbols) $\Gamma^\mu_{\alpha\beta}$ while the other one (13) contains only the derivatives of the metric g.

As described in Section 2.2, the metric of the mean space-time associated to a statistical ensemble of space-times Σ is simply the average of the metrics associated to the various members of Σ. On the other hand, the Christoffel symbols of the mean connection are indeed related to the averages of the Christoffel symbols of the space-times in Σ, but in no simple and direct way. Equation (13) is thus best suited for studying how the averaging procedure introduced in Section 2 works on geodesics.

4 Geodesics of the mean space-time as statistical averages

Let us consider again (see Sect. 2) an ensemble Σ of Einstein space-times $\mathcal{M}(\omega)$. Each space-time $\mathcal{M}(\omega)$ is endowed with a metric $g(\omega)$ and the associated connection $\Gamma(\omega)$, defined by (2).

Our goal is to relate, by a suitable averaging procedure, the geodesics of the mean space-time to the geodesics of the various space-times in Σ.

4.1 Averaging a collection of geodesics

Equations (11) and (13) make clear that, in any given space-time, an arbitrary geodesic $x(\lambda)$ is completely determined, at least locally, by a point through which it passes and the value of the 'velocity' $dx/d\lambda$ along the geodesic at that point.

We will therefore proceed as follows. Let us pick an arbitrary point M, a neighborhood $\mathcal{N}$ of M, a chart $\mathcal{C}$ defined on $\mathcal{N}$, common to all elements $\mathcal{M}(\omega)$ of the statistical ensemble Σ, and let X be the coordinates of M in $\mathcal{C}$.

Let also U be an arbitrary vector tangent to the space-time at M and let us focus on the geodesics which pass through M with 'velocity' U at point M.

Each of these geodesics can be represented, in the chart $\mathcal{C}$, by a set of 4 functions $x^\mu(\lambda,\omega)$ defined over, say $I \times \Omega$. By convention, the retained parametrization is supposed to be affine and the interval of variation of the parameter λ, denoted by I, can be taken without restriction to be common to all geodesics under consideration. Without any loss of generality, we further impose $\lambda = 0$ at $x = X$. The set of all these geodesics constitutes an M- and U-dependent collection of geodesics $\sigma_{M,U}$.

Each member $\gamma(\omega)$ of $\sigma_{M,U}$ verifies the geodesic equations (11) and (13):

$$p_\mu(\lambda,\omega) = g_{\mu\nu}(x(\lambda,\omega),\omega)\ \frac{\partial x^\nu}{\partial\lambda}_{|\lambda,\omega} \tag{14}$$

and

$$\frac{\partial p_\mu}{\partial\lambda}_{|\lambda,\omega} = \frac{1}{2}\ \partial_\mu g_{\alpha\beta}(\omega)_{|x(\lambda,\omega)}\ \frac{\partial x^\alpha}{\partial\lambda}_{|\lambda,\omega}\ \frac{\partial x^\beta}{\partial\lambda}_{|\lambda,\omega}. \tag{15}$$

One can construct two generally different curves which one might be tempted to consider, at least in the retained coordinate system, as averages of the geodesics in $\sigma_{M,U}$. Being averages, these two curves will be treated as curves in the mean space-time $\bar{\mathcal{M}}$. The first average curve, $\chi_\mathcal{C}(\lambda)$, is defined (in the chart $\mathcal{C}$) by:

$$\chi^\mu_\mathcal{C}(\lambda) = \langle x^\mu(\lambda,\omega)\rangle \tag{16}$$

and the associated 'momentum' $\varpi(\lambda)$ is:

$$\varpi_\mu(\lambda) = \bar{g}_{\mu\nu}(\chi_\mathcal{C}(\lambda))\ \frac{d\chi^\mu_\mathcal{C}}{d\lambda}. \tag{17}$$

As before, the brackets $\langle\cdots\rangle$ indicate statistical averaging over ω; in (16), the averaging is therefore done at fixed value of λ. If the probability measure on Ω is denoted by dp_ω, (16) simply means:

$$\chi^\mu_\mathcal{C}(\lambda) = \int_\Omega x^\mu(\lambda,\omega)\ dp_\omega. \tag{18}$$

Note also that (16) implies:

$$\frac{d\chi^\mu_\mathcal{C}}{d\lambda} = \left\langle \frac{\partial x^\mu(\omega)}{\partial\lambda}\right\rangle \tag{19}$$

Because of (16) and (19), the 'initial' conditions for $\chi_{\mathcal{C}}$ are:

$$\chi^\mu_{\mathcal{C}}(0) = X^\mu \tag{20}$$

and

$$\frac{d\chi^\mu_{\mathcal{C}}}{d\lambda}_{|0} = U^\mu. \tag{21}$$

Equation (17) then implies:

$$\varpi_\mu(0) = \bar{g}_{\mu\nu}(X)\ U^\nu. \tag{22}$$

As already mentioned in the Introduction (and as reflected in the very notation $\chi_{\mathcal{C}}$), the curve $\chi_{\mathcal{C}}$ depends, not only on the geodesics in $\sigma_{M,U}$, but also on the chart used in definition (16). In particular, $\chi_{\mathcal{C}}$ can therefore *not* be considered as an 'intrinsic average' of the geodesics in the collection $\sigma_{M,U}$, since different choices of $\mathcal{C}$ lead to different curves $\chi_{\mathcal{C}}$. Note however that a given curve $\chi_{\mathcal{C}}$, once defined by (16) within the chart $\mathcal{C}$, can also be described in other charts by parametric equations obtained from the expression of the functions $\chi^\mu_{\mathcal{C}}(\lambda)$ by the appropriate coordinate changes.

Physically, $\chi_{\mathcal{C}}$ represents to any observer using the chart $\mathcal{C}$ a natural average of the geodesics in the collection $\sigma_{M,U}$. The preceding remark on the non intrinsic character of definition (16) just means that different observers would then disagree on the definition of the averaging procedure.

The second 'average' curve $\xi_{\mathcal{C}}(\lambda)$ is defined through its 'momentum' components $\pi_\mu(\lambda)$ in the chart $\mathcal{C}$:

$$\pi_\mu(\lambda) = \langle p_\mu(\lambda,\omega)\rangle \tag{23}$$

and $\xi_{\mathcal{C}}(\lambda)$ is then fixed by:

$$\pi_\mu(\lambda) = \bar{g}_{\mu\nu}(\xi_{\mathcal{C}}(\lambda))\ \frac{d\xi^\nu_{\mathcal{C}}}{d\lambda} \tag{24}$$

and the 'initial' condition:

$$\xi^\mu_{\mathcal{C}}(0) = X^\mu. \tag{25}$$

As before, (23) simply means:

$$\pi_\mu(\lambda) = \int_\Omega p_\mu(\lambda,\omega)\ dp_\omega, \tag{26}$$

where dp_ω is the probability measure on Ω. Because of (23), one has:

$$\begin{aligned}\pi_\mu(0) &= \langle g_{\mu\nu}(X,\omega)U^\nu\rangle \\ &= \bar{g}_{\mu\nu}(X)U^\nu\end{aligned} \tag{27}$$

and (25) and (24) then imply:

$$\frac{d\xi^\mu_{\mathcal{C}}}{d\lambda}_{|0} = U^\mu. \tag{28}$$

As $\chi_{\mathcal{C}}$, the curve $\xi_{\mathcal{C}}$ does depend on the chart $\mathcal{C}$ used in definitions (23) and (24); in other words, $\xi_{\mathcal{C}}$ is not an intrinsic object. To an observer who uses the coordinate system $\mathcal{C}$, the curve $\xi_{\mathcal{C}}$ represents a second average of the geodesics in $\sigma_{M,U}$, alternate to $\chi_{\mathcal{C}}$. And, like $\chi_{\mathcal{C}}$, the curve $\xi_{\mathcal{C}}$, once defined in $\mathcal{C}$ by equations (23) and (24), can also be described in other charts by standard parametric equations obtained from the expression of the functions $\xi^\mu_{\mathcal{C}}(\lambda)$ by the appropriate coordinate changes.

Let us end this section by stressing that the two curves $\chi_{\mathcal{C}}(\lambda)$ and $\xi_{\mathcal{C}}(\lambda)$ are generically distinct, as can be seen by the following, very simple argument:

$$\begin{aligned}\pi_\mu(\lambda) &= \left\langle g_{\mu\nu}(x(\lambda,\omega),\omega)\ \frac{\partial x^\nu}{\partial\lambda}_{|\lambda,\omega}\right\rangle \\ &\neq \langle g_{\mu\nu}(\langle x(\lambda,\omega)\rangle,\omega)\rangle \left\langle\frac{\partial x^\nu}{\partial\lambda}_{|\lambda,\omega}\right\rangle = \varpi_\mu(\lambda).\end{aligned} \tag{29}$$

However, by equations (22) and (27), $\pi_\mu(0) = \varpi(0)$. Both curves therefore pass through point M, not only with the same 'velocities', but also with the same 'momenta'.

The reader is referred to Appendix A for covariant definitions of $\chi_{\mathcal{C}}$ and $\xi_{\mathcal{C}}$.

4.2 Link with the geodesics of the mean space-time

Let us now introduce the geodesic $\bar{\gamma}$ of the mean-metric $\bar{g}$ which passes through point M with 'velocity' U for $\lambda = 0$. We will now show that $\bar{\gamma}$ provides, around point M, a first order approximation to all curves $\chi_{\mathcal{C}}$ and a second order approximation to all curves $\xi_{\mathcal{C}}$. The geodesic $\bar{\gamma}$ of the mean gravitational field can thus be considered as *the* natural, intrinsically defined local average of the geodesics in the collection $\sigma_{M,U}$.

All the following calculations will be carried out in a single but arbitrary chart $\mathcal{C}$ defined around point M and the parametric equation of the geodesic $\bar{\gamma}$ in $\mathcal{C}$ will be denoted by $\bar{x}(\lambda)$.

For any smooth curve $y(\lambda)$ of the mean space-time which passes through M at $\lambda = 0$ with 'velocity' U, one can write:

$$y^\mu(\lambda) = X^\mu + \lambda U^\mu + \ \frac{1}{2}\ \lambda^2\ \frac{d^2y^\mu}{d\lambda^2}_{|0} + O(\lambda^3). \tag{30}$$

Let $q(\lambda)$ be the 'momentum' along the curve $y(\lambda)$:

$$q_\mu(\lambda) = \bar{g}_{\mu\nu}(y(\lambda))\ \frac{dy^\nu}{d\lambda}. \tag{31}$$

Differentiating (31) with respect to λ, one obtains:

$$\frac{dq_\mu}{d\lambda} = \partial_\alpha\bar{g}_{\mu\nu|y(\lambda)}\ \frac{dy^\nu}{d\lambda}\frac{dy^\alpha}{d\lambda} + \bar{g}_{\mu\nu}(y(\lambda))\ \frac{d^2y^\nu}{d\lambda^2} \tag{32}$$

which leads, for $\lambda = 0$, to:

$$\frac{d^2y^\mu}{d\lambda^2}_{|0} = \bar{g}^{\mu\nu}(X)\left[\frac{dq_\nu}{d\lambda}_{|0} - \partial_\alpha\bar{g}_{\nu\beta|X}U^\alpha U^\beta\right]. \tag{33}$$

By (30) and (33), the second order behaviour of $y(\lambda)$ around $\lambda = 0$ (point M) is therefore entirely fixed by the value of $dq/d\lambda$ at $\lambda = 0$. Let us evaluate this quantity for both $\xi_{\mathcal{C}}(\lambda)$ and $\bar{x}(\lambda)$.

As far as $\xi_{\mathcal{C}}(\lambda)$ is concerned, equations (23) and (15) imply directly:

$$\frac{d\pi_\mu}{d\lambda}_{|0} = \left\langle \frac{\partial p_\mu}{\partial \lambda}_{|0,\omega} \right\rangle = \frac{1}{2}\, \partial_\mu \bar{g}_{\alpha\beta|X} U^\alpha U^\beta. \tag{34}$$

Now, by definition of geodesics, the 'momentum' $\bar{p}(\lambda)$ along the curve $\bar{x}(\lambda)$ verifies the equation:

$$\frac{d\bar{p}_\mu}{d\lambda} = \frac{1}{2}\, \partial_\mu \bar{g}_{\alpha\beta|\bar{x}(\lambda)} \frac{d\bar{x}^\alpha}{d\lambda}\, \frac{d\bar{x}^\beta}{d\lambda}. \tag{35}$$

This leads to:

$$\frac{d\bar{p}_\mu}{d\lambda}_{|0} = \frac{1}{2}\, \partial_\mu \bar{g}_{\alpha\beta|X} U^\alpha U^\beta, \tag{36}$$

which, together with (34), proves that:

$$\frac{d\bar{p}_\mu}{d\lambda}_{|0} = \frac{d\pi_\mu}{d\lambda}_{|0}. \tag{37}$$

One therefore has, by equations (30) and (33):

$$\bar{x}^\mu(\lambda) - \xi^\mu_{\mathcal{C}}(\lambda) = O\left(\lambda^3\right). \tag{38}$$

Since the chart $\mathcal{C}$ has been kept arbitrary, the geodesic $\bar{x}(\lambda)$ is locally (around point M) a second-order approximation (in λ) to all the curves $\xi_{\mathcal{C}}(\lambda)$.

On the other hand, (29) shows that, generically, the first derivatives of π and ϖ at $\lambda = 0$ do *not* coincide, even though $\pi(0) = \varpi(0)$. Thus, generically,

$$\bar{x}^\mu(\lambda) - \chi^\mu_{\mathcal{C}}(\lambda) = O\left(\lambda^2\right) \tag{39}$$

and

$$\xi^\mu_{\mathcal{C}}(\lambda) - \chi^\mu_{\mathcal{C}}(\lambda) = O\left(\lambda^2\right) \tag{40}$$

for all charts $\mathcal{C}$. The geodesic $\bar{x}(\lambda)$ is therefore only a first-order approximation to the curves $\chi_{\mathcal{C}}(\lambda)$ and the two curves $\chi_{\mathcal{C}}(\lambda)$ and $\xi_{\mathcal{C}}(\lambda)$ also differ by second-order terms.

The validity of equation (38) for all charts $\mathcal{C}$ (and, to a lesser degree, the validity of Eq. (39)) makes it natural to define the intrinsic average of the geodesics in the collection $\sigma_{M,U}$ as the geodesic of the mean space-time $\bar{\gamma}$ which passes through M with 'velocity' U.

Since both X and U have so far been kept arbitrary, this conclusion can be restated in the following, perhaps more illuminating way. Let us pick an arbitrary geodesic $\bar{\gamma}$ of the mean space-time $\bar{\mathcal{M}}$ and a point M on this geodesic. Let U be the 'velocity' at point M along this geodesic (for a given affine parametrization). Then, around point M, $\bar{\gamma}$ is the natural, intrinsically defined average of all the geodesics of the 'unaveraged' space-times $\mathcal{M}(\omega)$ which pass through point M with 'velocity' U.

The locally defined average of the collection of geodesics $\sigma_{M,U}$ is thus a geodesic of the mean space-time and, conversely, any given geodesic in $\bar{\mathcal{M}}$ appears as local average of geodesics of the various space-times $\mathcal{M}(\omega)$.

5 Discussion

This section is organized as follows. The first Section (5.1) comments upon the final statement of Section 4.2 by adding to the material hitherto presented a discussion of the respective time-like, space-like or null character of the various geodesics involved in the averaging procedure. Sections 5.2 and 5.3 are devoted to astrophysical and cosmological problems directly connected with the general issue of the present article. Section 5.2 provides some rough calculations supporting the view that the mean-field approximation does furnish reasonable results on the cosmological scale. Section 5.3 discusses how the Sachs equations model the evolution of a beam of photons and justifies the use of the mean-field approximation in these equations.

5.1 Respective character of the various geodesics

Let us elaborate on the time-like, space-like or null character of the involved geodesics. This character is entirely determined by the vector U. But whether U is time-like, space-like or null depends on the metric one uses to evaluate U^2. As a consequence, the geodesic $\bar{\gamma}$ being, say, time-like does not necessarily imply that all geodesics $\gamma(\omega)$ are also time-like. One can nevertheless write:

$$\bar{g}_{\mu\nu}(X)\; U^\mu U^\nu = \langle g_{\mu\nu}(X,\omega) U^\mu U^\nu \rangle, \tag{41}$$

which only allows the following three, very general conclusions. First, if U is time-like (resp. space-like) in $\bar{\mathcal{M}}$ at X, there is then at least one of the $\mathcal{M}(\omega)$ in which U is time-like (resp. space-like) at X. Second, if U is time-like, space-like or null at X in all the space-times $\mathcal{M}(\omega)$, it is then also time-like, space-like or null at X in $\bar{\mathcal{M}}$. Finally, if U is null in $\bar{\mathcal{M}}$ at X, then U is either null in all space-times of the statistical ensemble Σ, or Σ contains space-times for which U is time-like together with space-times for which U is space-like.

This has some important physical consequences. Free general relativistic point-particles move on time-like or null geodesics. The preceding remark shows that a system may behave as a point-particle on large scale (i.e. in the mean space-time $\bar{\mathcal{M}}$) but not on smaller scales (i.e. in members $\mathcal{M}(\omega)$ of the statistical ensemble of space-times which is used to define $\bar{\mathcal{M}}$). In other words, the concept of point-like (test-)particle is not scale-invariant. This echoes the general conclusion that the equation of state for matter is changed upon averaging the gravitational field [3].

On the other hand, a system which behaves as a point-like particle on some scale will also behave locally as a point-like particle on all larger scales.

The very special case of ensembles of conformal metrics deserves particular mention. Suppose all the metrics $g(\omega)$ are conformal to one another; in that case, one can pick up one of these metrics, say $\mathring{g} = g(\omega^0)$, and write for all $\omega \in \Omega$:

$$g(\omega) = F^2(\omega)\mathring{g}, \tag{42}$$

where $F(\omega)$ is the so-called conformal factor relating $g(\omega)$ to $\mathring{g}$. One has then:

$$\bar{g} = \langle g(\omega) \rangle = \left\langle F^2(\omega) \right\rangle \mathring{g} \\ = \bar{F}^2 \mathring{g}, \tag{43}$$

which defines $\bar{F}$ and also shows that $\bar{g}$ is conformal to $\mathring{g}$ and, therefore, to all the metrics $g(\omega)$.

Now, metrics conformal to one another share the same light-cone structure [15] and any vector which is time-like, space-like or null for one of these metrics is also time-like, space-like or null for the other metrics. Thus, in case the statistical ensemble is made of metrics conformal to one another, the mean metric is conformal to all the metrics in the ensemble and, therefore, any geodesic $\bar{\gamma}$ of $\bar{\mathcal{M}}$ shares its time-like, space-like or null character with all the geodesics $\gamma(\omega)$ of which it is the local average.

5.2 Orders of magnitude: what does 'locally' mean in the cosmological context?

As insisted upon in Section 4, the interpretation of the geodesics of the mean metric as averages of the geodesics of the fluctuating field is only local, in the mathematical sense of the word. But what does 'local' mean in practical astrophysical or cosmological applications? In particular, is the mean field approximation justified on scales comparable to the Hubble length? This apparently innocuous question is actually very difficult to answer in its full generality. Considering however the importance of the issue, it has been deemed necessary to include a preliminary discussion of it in this article. What follows is only intended to give a flavor of what tackling the problem entails; the results to be presented also suggest that the mean-field approximation *is* reasonable in the cosmological context, but the arguments of this section can by no means be considered fully rigorous. It is hoped that their presentation will serve as an incentive to further, perhaps numerical study and, *de facto*, to a complete treatment and validation (or invalidation) of the mean field approximation in studying geodetic motions on the cosmological scale.

Let us consider a photon which is detected on Earth, at point E of space-time, after having travelled through a temporally and spatially fluctuating gravitational field. Let V be the detected 'velocity' of the photon at point E. To reconstruct the trajectory of the photon before its detection, one needs a statistical ensemble of space-times which correctly models the fluctuating field and which takes into account the fact that a photon was 'detected' at point E with velocity V. Such a statistical ensemble will typically be represented by an ensemble of metrics $g(\omega)$ which all coincide at point E and for which V is null at point E:

$$g_{\mu\nu}(X_E, \omega) V^\mu V^\nu = 0. \tag{44}$$

Investigating precisely the motion of the photon in such an ensemble, even for very special forms of the metrics $g(\omega)$, is extremely difficult and can best be achieved by extensive numerical simulations.

However, if one's primary interest lies in knowing only if the mean field approximation does (or does not) remain valid for motions over cosmological scales, it is fortunately possible to replace the preceding problem by a slightly more general one, which, thanks to its more general character, is at least susceptible of a rough analytical treatment.

The idea is simply to replace the point E in the preceding problem by an arbitrary point M where all metrics of the ensemble of space-times do *not* necessarily coincide and to investigate 'how fast' the trajectories which originate from point M with a common 'velocity' U diverge from the trajectory in the mean field. Naturally, since all metrics of the statistical ensemble coincide at point E but generally do not at point M, one expects the trajectories that pass through M with a given 'velocity' U to diverge faster than those which pass through E (with a common 'velocity' V not necessarily equal to U).

In the notations of the preceding sections, the set of geodesics corresponding to the first, 'physical' problem is $\sigma_{E,V}$ and the set of geodesics corresponding to the more general problem is $\sigma_{M,U}$. According to the remark above, the validity of the mean field approximation is more stringently tested on $\sigma_{M,U}$ than on $\sigma_{E,V}$. Consequently, if the mean field approximation is shown to be valid over cosmological distances for the geodesics of $\sigma_{M,U}$, one can reasonably think it will also be valid for the geodesics of $\sigma_{E,V}$ i.e. in the real, 'physical' problem.

The final aim of this section is to present some order of magnitude estimates that support the mean field approximation over cosmological distances for the geodesics of $\sigma_{M,U}$.

To proceed, let us introduce an ensemble Σ of space-times (metrics) which adequately models, in the cosmological context, the fluctuations of the gravitational field. In a chart $\mathcal{C}$ common to all members of Σ, the metric $g(x, \omega)$ of $\mathcal{M}(\omega)$ takes the form [4,13]:

$$ds^2_\omega = a^2(t) \left(\eta_{\alpha\beta} + h_{\alpha\beta}(x, \omega) \right) dx^\alpha dx^\beta \tag{45}$$

where η stands for the Minkovski metric, $a(t)$ is the expansion factor of the universe and $h(x, \omega)$ represents the perturbation to the homogeneous, isotropic, spatially flat model characterized by $a(t)$. Observational cosmology seems to indicate that only a very small fraction of the physical space-time is possibly endowed with large curvature fluctuations [13]. This translates into $\mid h \mid = O(\epsilon) \ll 1$, where the vertical bars denote a suitable norm on the function space to which $h(., \omega)$ belongs. We also suppose that, by construction, the ensemble Σ verifies $\langle h(x, \omega) \rangle = 0$ for all x; this ensures that the average metric $\bar{g}$ does indeed correspond to the homogeneous isotropic model determined by the expansion factor a: $\bar{g}_{\mu\nu} = a^2(t)\eta_{\mu\nu}$. The ω-dependent perturbation $\delta g_{\mu\nu}(\omega)$ to this mean metric is simply $\delta g_{\mu\nu}(\omega) = a^2(t) h_{\mu\nu}(\omega)$ and $\mid \delta g \mid / \mid g \mid = O(\epsilon)$. The coordinate t is sometimes called the conformal time. Its relation to the so-called comoving time widely used in cosmology is given by equation (54) below.

We further impose on Σ a condition similar to (44):

$$g_{\mu\nu}(X, \omega) U^\mu U^\nu = 0 \tag{46}$$

for all $\omega \in \Omega$. This ensures that, in each space-time of the ensemble, the trajectory of the photon is a null geodesics of the metric associated to this space-time. As a consequence, U is also null for the average metric and therefore, in the chart $\mathcal{C}$, $U^0 = \sqrt{\mathbf{U}^2}$, (where $\mathbf{U}^2$ stands for the standard Euclidean square of $\mathbf{U}$). Finally, X will stand for the coordinates of point M in $\mathcal{C}$ and the affine parameter used along all the geodesics will be taken to vanish at point M.

The validity of the mean field approximation can be measured, in the working chart $\mathcal{C}$, by the two different functions of λ, $\Delta(\lambda) =\mid \bar{x}(\lambda) - \chi_{\mathcal{C}}(\lambda) \mid$ and $\delta(\lambda) =\mid \bar{x}(\lambda) - \xi_{\mathcal{C}}(\lambda) \mid$. It turns out that, around $\lambda = 0$ (i.e. around point M), the function Δ is much easier to evaluate than δ and this discussion will therefore concentrate on Δ. Note that, around $\lambda = 0$, $\Delta(\lambda) = O(\lambda^2)$ while $\delta(\lambda) = O(\lambda^3)$ (see Sect. 4.2); the limit of validity of the mean field approximation, as measured by Δ, is therefore more stringent than the validity limit measured by δ.

By (30),

$$\Delta(\lambda) = \frac{\lambda^2}{2} \left| \frac{d^2\bar{x}}{d\lambda^2} - \frac{d^2\chi_{\mathcal{C}}}{d\lambda^2} \right| + O\left(\lambda^3\right). \tag{47}$$

One can differentiate (14) and use (15) in the result to obtain:

$$\begin{aligned} \frac{\partial^2 x^\mu}{\partial\lambda^2} &= \frac{1}{2}\, g^{\mu\nu}(x(\lambda,\omega),\omega)\ \partial_\nu g_{\alpha\beta}(\omega)_{|x(\lambda,\omega)} \frac{\partial x^\alpha}{\partial\lambda}_{|\lambda,\omega} \frac{\partial x^\beta}{\partial\lambda}_{|\lambda,\omega} \\ &- g^{\mu\nu}(x(\lambda,\omega),\omega)\ \partial_\alpha g_{\nu\beta}(\omega)_{|x(\lambda,\omega)} \frac{\partial x^\alpha}{\partial\lambda}_{|\lambda,\omega} \frac{\partial x^\beta}{\partial\lambda}_{|\lambda,\omega}, \end{aligned} \tag{48}$$

which leads to:

$$\begin{aligned} \frac{d^2\chi^\mu_{\mathcal{C}}}{d\lambda^2}_{|0} &= \frac{1}{2}\ \left\langle g^{\mu\nu}(X,\omega)\ \partial_\nu g_{\alpha\beta}(\omega)_{|X}\right\rangle U^\alpha U^\beta \\ &- \left\langle g^{\mu\nu}(X,\omega)\ \partial_\alpha g_{\nu\beta}(\omega)_{|X}\right\rangle U^\alpha U^\beta. \end{aligned} \tag{49}$$

On the other hand, equations (33) and (35) imply:

$$\begin{aligned} \frac{d^2\bar{x}^\mu}{d\lambda^2}_{|0} &= \frac{1}{2}\ \bar{g}^{\mu\nu}(X,\omega)\ \partial_\nu \bar{g}_{\alpha\beta}(\omega)_{|X} U^\alpha U^\beta \\ &- \bar{g}^{\mu\nu}(X,\omega)\ \partial_\alpha \bar{g}_{\nu\beta}(\omega)_{|X} U^\alpha U^\beta. \end{aligned} \tag{50}$$

One thus finds:

$$\begin{aligned} \frac{d^2\bar{x}^\mu}{d\lambda^2}_{|0} - \frac{d^2\chi^\mu_{\mathcal{C}}}{d\lambda^2}_{|0} &= U^\alpha U^\beta \\ \times \Big\{ \frac{1}{2}\ &\left(\bar{g}^{\mu\nu}(X)\ \partial_\nu \bar{g}_{\alpha\beta|X} - \left\langle g^{\mu\nu}(X,\omega)\ \partial_\nu g_{\alpha\beta}(\omega)_{|X}\right\rangle\right) \\ &- \left(\bar{g}^{\mu\nu}(X)\partial_\alpha \bar{g}_{\nu\beta|X} - \left\langle g^{\mu\nu}(X,\omega)\ \partial_\alpha g_{\nu\beta}(\omega)_{|X}\right\rangle\right)\Big\}. \end{aligned} \tag{51}$$

The mean-square displacements appearing on the right-hand side of this equation can be put into a somewhat more telling form for ensembles of metrics which, like Σ, describe small perturbations to a certain mean gravitational field. Indeed, for any metric $g_{\mu\nu}(x,\omega) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x,\omega)$, $\mid \delta g \mid / \mid g \mid = O(\epsilon) \ll 1$, a straightforward calculation shows that, generically, the components $g^{\mu\nu}$ of the inverse to $g_{\mu\nu}$ can be written as:

$$\begin{aligned} g^{\mu\nu}(x,\omega) &= \bar{g}^{\mu\nu}(x) - \bar{g}^{\mu\alpha}(x)\bar{g}^{\nu\beta}(x)\ \delta g_{\alpha\beta}(x,\omega) \\ &+ \bar{g}^{\mu\alpha}(x)\bar{g}^{\nu\rho}(x)\bar{g}^{\beta\sigma}(x)\ \delta g_{\alpha\beta}(x,\omega)\delta g_{\rho\sigma}(x,\omega) + O\left(\epsilon^3\right). \end{aligned} \tag{52}$$

Applying this result to (45), one gets the following, more explicit expression of the first mean square displacement appearing on the right-hand side of (51):

$$\begin{aligned} \frac{U^\alpha U^\beta}{2}\ &\left(\bar{g}^{\mu\nu}(X)\ \partial_\nu \bar{g}_{\alpha\beta|X} - \left\langle g^{\mu\nu}(X,\omega)\ \partial_\nu g_{\alpha\beta}(\omega)_{|X}\right\rangle\right) = \\ &\frac{U^\alpha U^\beta}{2}\ \Bigg(h^{\mu i}(X)\partial_i h_{\alpha\beta|X} + \frac{1}{c}\ h^{\mu 0}(X) \\ &\times \left(2\ \frac{\dot{a}}{a}(X)\ h_{\alpha\beta}(X) + \partial_t h_{\alpha\beta|X}\right)\Bigg), \end{aligned} \tag{53}$$

where the indices are raised on h by the Minkovski metric η and $\dot{a} = da/dt$. The Latin indices in (53) designate the 'spatial' space-time variables.

An expression similar to (53) can be obtained for the second mean square displacement in (51). The detailed analysis of that second contribution will not be considered here because it confirms the conclusions obtained from the sole consideration of (53).

We are primarily interested in the scaling properties of expression (53). Since t is the so-called conformal time, the quotient $\dot{a}/a$ is *not* the Hubble constant $H(t)$ at time t. Indeed, the comoving time t_c is related to the conformal time t by:

$$dt_c^2 = a^2(t)dt^2. \tag{54}$$

One thus has:

$$\frac{\dot{a}}{a}_{|t} = a(t)H(t). \tag{55}$$

Let now L_x be the typical size (variation scale) of the perturbation h, expressed in the chart $\mathcal{C}$. The *physical* variation scale of h is then $L_p = a(t)L_x$. The right-hand side of (53) contains two types of contributions: the first and third terms scale as $\mid U \mid^2 h^2 L_x^{-1}$, while the second term scales as $\mid U \mid^2 h^2 aH/c$. These contribute to $\Delta(\lambda)$ (see (47)) two terms, which scale respectively as $\lambda^2 \mid U \mid^2 h^2 L_x^{-1}$ and $\lambda^2 \mid U \mid^2 h^2 aH/c$. Note that the second and third terms on the right-hand side of (53) scale differently because, according to (55), the typical variation scale of the expansion factor $a(t)$ is $a(t)H(t)$, which is a priori different from the temporal variation scale of the perturbation h.

By definition of U, $\lambda \mid U \mid$ scales as D_x, the 'distance' travelled by the photon during a variation λ of the parameter, expressed in coordinates x. The corresponding 'physical distance' is $D_p = a(t)D_x$. The function Δ thus contains a contribution Δ_1 which scales as ${D_x}^2 h^2 L_x^{-1}$ and another one, Δ_2, which scales as ${D_x}^2 h^2 aH/c$.

It seems reasonable to evaluate the validity of the mean-field approximation on geodetic motions by comparing Δ to the 'distance' D_x. In units D_x, Δ_1 scales as $D_x L_x^{-1} h^2 = D_p L_p^{-1} h^2$ and Δ_2 scales as $D_x a H h^2/c = D_p H h^2/c$. The mean field approximation is a priori good as long as these quantities remain much smaller than one. Conversely, the mean field approximation probably breaks down after the photon has travelled a physical distance which makes Δ_1 or Δ_2 of order one. One thus finds, with obvious notations:

$$D_{p1}^{lim} = \frac{1}{h^2} \, L_p \tag{56}$$
$$D_{p2}^{lim} = \frac{1}{h^2} \, cH^{-1}.$$

Since $h \ll 1$, the distance D_{p2}^{lim} can typically not be reached in the Hubble time. Let us therefore concentrate the discussion on D_{p1}^{lim}.

Given the typical size L_p of a perturbation, the associated typical mass μ can be used to evaluate a typical gravitational potential per unit mass $\phi = G\mu/L_p$, which corresponds to a typical perturbation of the metric $h = \phi/c^2 = G\mu/L_p c^2$. One thus finds:

$$D_{p1}^{lim} = \frac{L_p^3 c^4}{G^2 \mu^2} \tag{57}$$

or, equivalently:

$$D_{p1}^{lim} \, \frac{H}{c} \; = L_p \, \frac{H}{c} \, \frac{L_p^2 c^4}{G^2 \mu^2} \tag{58}$$

The following table gives approximate values for $D_{p1}^{lim} H/c$ for typical structures. These values have been obtained with $cH^{-1} = cH_0^{-1} = 3000$ Mpc and the orders of magnitude for L_p and μ are taken from [12].

Structure	L_p	μ	$D_{p1}^{lim} H/c$
Sun	7×10^{10} cm	$2 \times 10^{33}\mathrm{g} \equiv M_\odot$	2×10^{-8}
Galaxy	15 kpc	$10^{11}\ M_\odot$	6×10^5
Cluster	5 Mpc	$10^{13}\ M_\odot$	2×10^9
Supercluster	50 Mpc	$10^{15}\ M_\odot$	2×10^8

The obtained values strongly suggest that large-scale structures such as galaxies, clusters and superclusters do not deviate photons on cosmological scale from the trajectories they would have in the idealized, homogeneous and isotropic cosmological models[4]. On the other hand, the result obtained for the Sun does suggest that small-scale structures indeed deviate photons on cosmological scale. This type of lensing needs only be considered in the 'rare' cases when such a small-scale structure is very close to the line of sight of the observed source.

The results of this section thus confirm the global applicability of the mean field approximation to the study of motion over the cosmological scale. Naturally, as indicated earlier, this conclusion definitely needs to be confirmed by a more systematic, for example numerical analysis.

5.3 On using the mean field approximation for studying the evolution of a beam of photons

Observing a source S essentially comes down to detecting on Earth particles, let us say photons, emitted by S. These photons form a 'narrow' beam, i.e. there is a 'not too important' spread in the emission direction of these photons. One of the first problems in interpreting observations is therefore to model the propagation, not (only) of a single photon, but (also) of 'narrow' beams of photons in the physical space-time. This connects naturally with the material presented in this article (in particular in Sect. 5.2) and in [3]. We now wish to elaborate on this connection. With the hope to make the following discussion as general and, therefore, as enlightening as possible, it has been decided to review here how both the general mean field theory introduced in [3] and the developments of the preceding sections articulate with the 'real' astrophysical problematics.

Let us consider a beam of photons emitted by a source S with a certain spread in the emission direction and propagating towards Earth in a given gravitational field represented by the metric g. There are two 'basic assumptions' behind the standard analysis [13] of the propagation of the beam. The first 'assumption' is that each single photon in the beam propagates along a null geodesic of the gravitational field g, this geodesic being completely determined by the frequency and direction emission of the considered photon at its source (or by its frequency and direction at any point of its trajectory, including the detector on Earth). The second 'assumption' exploits the a priori 'narrow' character of the beam. The standard reasoning goes as follows: If the beam is 'sufficiently narrow', the geodesics followed by the various photons are 'sufficiently close' to each other and the time evolution of the spread of the beam can be realistically evaluated through the so-called geodesic deviation equation [15]. Of course, the expressions 'sufficiently narrow' and 'sufficiently close' essentially mean that, in a local coordinate system attached to a point located on the trajectory of the beam [13], the typical size of the beam is much smaller than the typical variation scale of the gravitational field. The analysis then yields a couple of differential equations for the evolution of the expansion and shear tensor of the beam [14]. These equations, together with some of their possible applications, are discussed in [13].

Let us now take into account the fact that we as observers do not know with absolute precision the gravitational field in which the beam propagates. Indeed, a real gravitational field is generally fluctuating, both spatially and temporally, and we often have, at best, indications of

[4] Of course, the expansion of Δ in powers of λ around point M probably ceases to be valid for values of D_{p1} comparable to (or much greater than) the Hubble length. The precise values of D_{p1}^{lim} in the table should therefore not be taken too seriously; but the orders of magnitude *do* support the validity of the mean field approximation over cosmological distances.

its mean value and, possibly, of its mean square displacements. It is therefore common to replace, in the afore presented analysis, the real metric g by a metric $\bar{g}$ associated to an average or mean gravitational field, with the hope that the results about the time evolution of the expansion and the shear tensor of the beam will be at least statistically correct, the statistics being made on a large number of different beams.

For this to be true, both assumptions made in deriving the Sachs equations have to remain at least statistically valid if one replaces the real fluctuating gravitational field by the average or mean gravitational field. Focusing for the moment on the first assumption only, we are led to conclude that using the mean gravitational field in the Sachs equations will only yield reasonable results if, statistically speaking, the null geodesics of the mean gravitational field can be assimilated to the null geodesics of the real fluctuating field. In other words, a necessary condition for the mean field approximation to yield physically realistic results when applied to Sachs equations is that, on the average, the motions of single photons in the fluctuating field be well approximated by motions in the mean field.

To check if this condition applies, it is obviously necessary to have a precise definition of the mean gravitational field in terms of the unaveraged one. It is at this point that the results presented in [3] enter the discussion. Since the equations of general relativity are non linear, the precise delineation of the very concept of mean gravitational field is non trivial. This problem has been addressed in [3] within a purely statistical approach. This is clearly the most general approach possible, and more classical temporal and spatial averages can always be recast as statistical averages [3]. The main result of [3] can be stated as follows: the only way to ensure the mean gravitational field does obey the equations of general relativity[5] is to take its metric $\bar{g}$ to be the average of the metrics $g(\omega)$ over the statistical ensemble of space-times used to represent the (unaveraged) fluctuating gravitational field.

Whereas [3] dealt with the problem of defining what a mean gravitational field is, the present article establishes (in Sect. 4) a local correspondence between the geodetic motions in such a mean gravitational field and the average geodetic motions in the space-times of the statistical ensemble used to construct the mean field. Succinctly stated, given a statistical ensemble Σ of space-times which represents a fluctuating gravitational field, all geodesics of the mean field can be viewed as statistical averages of geodesics belonging to the various members of the ensemble. Somewhat loosely speaking, the averages of the geodesics of the fluctuating field are identical to geodesics of the mean field. As repeatedly emphasized earlier, the preceding statement is only local (in the mathematical sense of the word) but the results of Section 5.2 strongly suggest that its applicability actually extends, in cosmology, up to the Hubble length[6].

[5] If the original unaveraged field does.

[6] If one neglects the 'rare' events for which a very small 'structure' like a star is sufficiently close to the line of sight to induce a strong lensing.

Thus, the results presented in this article justify at least locally, and probably up to scales comparable to the Hubble-length, the first of the two conditions necessary for the use of the mean gravitational field in Sachs equations.

Now to the second condition. Let us suppose that Sachs equations realistically describe the evolution of the beam in the unaveraged gravitational field. As already elaborated upon, this essentially means that the spread of the beam is much smaller than the typical variation scale of the unaveraged field. Since the typical variation scale of the *averaged* field is necessarily larger than the variation scale of the unaveraged one, the spread of the beam is then also much smaller than the variation scale of the *averaged* field. And this is precisely the second condition which warrants that the combined use of the mean gravitational field and of the Sachs equations describe properly the average spreading and shearing of the photon beam.

The work presented in [3] and in the preceding sections of the present article thus fully justifies the mean field approximation in Sachs equations.

6 Summary and conclusion

A recent work [3] has given a precise meaning to the mean field approximation in general relativity. The aim of the present article was to investigate if the geodesics of the mean field could be considered as averages of the geodesics of the unaveraged field. To make this article as self-contained as possible, the precise construction of a mean space-time associated to a given statistical ensemble of space-times has been reviewed in Section 2. In Section 3, I have presented in detail the two standard possible definitions of geodesics in Einstein space-times, devoting more attention to the second, variational definition because it is ideally suited to an in depth study of the working of the averaging procedure introduced in Section 2.

Section 4 contains the main result of the article. Succinctly stated, it has been shown that any (given) geodesic of the mean space-time can be considered locally as the average of a collection of geodesics, each geodesic in the collection belonging to one of the space-times in the statistical ensemble under consideration. This result has been further discussed in Section 5, where related astrophysical and cosmological issues have also been addressed. In particular, strong arguments have been given to support mean field theory as a realistic tool to describe geodetic motions on the scale of the Hubble length and the use of the mean field approximation in Sachs optical equations has been justified.

Let me end this article by mentioning some important extensions of the work which has been presented here. First of all, the validity of the mean field approximation on cosmological scales should be investigated in full detail, for example via computer simulations, if only to make the conclusions reached in Section 5 of this article stand on firmer ground. On the contrary, one can wonder on how misleading the mean-field approximation would be when applied to gravitational fields with fluctuations of 'large' amplitude and/or 'high' frequency.

In a more general direction, is it possible to extend the results presented in this article to situations where the topology of the space-time is not fixed, but is itself averaged upon? This promises to be a most difficult but hopefully very rewarding step and its completion might open new vistas on several fields of great theoretical interest, such as early universe cosmology or quantum field theory in curved space-time.

It is a pleasure to thank both (anonymous) referees whose various suggestions definitely led to a substantial improvement of the article.

Appendix A

Terminology

Given a manifold equipped with a metric g and the Levi-Civita connection $\Gamma(g)$ of that metric, an object defined on that manifold will be called intrinsic if its definition involves only the metric g (and its various derivatives). For example, the curvature tensor of the Levi-Civita connection of the metric is an intrinsic quantity. On the contrary, the coordinates of a point or the components of a tensor field in a given chart $\mathcal{C}$ are not intrinsic quantities because their definitions involve the chart $\mathcal{C}$; to mention another example, the projector Δ onto the subspace orthogonal to a certain vector field U is not intrinsic either because the definition of Δ involves U.

By extension, given a statistical ensemble Σ of space-times $\mathcal{M}(\omega)$ (see Sect. 2.1), each $\mathcal{M}(\omega)$ being equipped with a metric $g(\omega)$ and the Levi-Civita connection $\Gamma(g(\omega))$ of that metric, an object (typically defined on the mean space-time $\bar{\mathcal{M}}$ associated to Σ) will be called intrinsic if its definition only involves the various metrics $g(\omega)$.

Now, we will say that a certain statement or a certain set of equations is covariant if the same statement or set of equations, once true in a certain chart, is also true in all other charts. For example, the statement '$R = 0$ at point M', where R is the scalar curvature of $\Gamma(g)$, is a covariant statement about an intrinsic object. On the contrary, '$R^{01} = 0$ at all points' is a non covariant statement about an intrinsic object; indeed, the Ricci tensor is intrinsically defined but the truth of the relation $R^{01} = 0$ depends on the coordinate system used on the manifold.

Let me end this section by giving examples of covariant and non covariant statements about non intrinsic quantities. Given a vector field U normed to unity, the relation $\Delta_{\mu\nu} = U_\mu U_\nu - g_{\mu\nu}$ can be viewed as a covariant definition of the projector Δ (which is a non intrinsic object) onto the subspace orthogonal to U. On the contrary, the statement '$\Delta_{00} = U_0 U_0 - 1$ at all points' is non covariant because it is only true in the charts where $g_{00} = 1$ at all points.

Covariant definitions of various curves considered in the article

This section of the Appendix offers some further comments on various possible definitions of the curves $\chi_{\mathcal{C}}$ and $\xi_{\mathcal{C}}$ introduced in Section 4.

As already mentioned several times, these curves are not intrinsic averages of the geodesics in the collection $\sigma_{M,U}$. In Section 4, $\chi_{\mathcal{C}}$ has been defined by equation (16) and $\xi_{\mathcal{C}}$ has been defined by (23) and (24). None of these equations, is covariant. For example, the mean value in (23) is obtained by simply adding the coordinate-basis components of vectors defined at different points in space time, and writing such a sum covariantly necessitates the explicit introduction of a connection (for example, through its coordinate-basis coefficients). Let us therefore investigate how the non intrinsically defined curves $\chi_{\mathcal{C}}$ and $\xi_{\mathcal{C}}$ can be given covariant definitions.

Let us start with $\chi_{\mathcal{C}}$. The chart $\mathcal{C}$ is associated to a bi-continuous mapping $\phi_{\mathcal{C}}$ defined from an open subset of the space-time which contains point M onto $\mathbb{R}^4$. Let $N(\lambda,\omega)$ be a running point on the geodesic $\gamma(\omega)$ and let $N_{\mathcal{C}}(\lambda)$ be a running point on $\chi_{\mathcal{C}}$.

A covariant definition of $\chi_{\mathcal{C}}$ reads:

$$N_{\mathcal{C}}(\lambda) = \phi_{\mathcal{C}}^{-1}\left(\langle \phi_{\mathcal{C}}\left(N(\lambda,\omega)\right)\rangle\right), \tag{59}$$

where $\phi_{\mathcal{C}}^{-1}$ is the inverse to $\phi_{\mathcal{C}}$ and

$$\langle \phi_{\mathcal{C}}\left(N(\lambda,\omega)\right)\rangle = \int_{\Omega} \phi_{\mathcal{C}}\left(N(\lambda,\omega)\right) dp_\omega, \tag{60}$$

dp_ω being the probability measure on Ω. Equation (60) makes mathematical sense because $\mathbb{R}^4$, unlike the space-time manifold, is also a vector space, the 'points' of which can therefore be added to one another or multiplied by real numbers.

Note also that (59) supposes that $\langle \phi_{\mathcal{C}}\left(N(\lambda,\omega)\right)\rangle$ belongs to the domain of $\phi_{\mathcal{C}}^{-1}$. In particular, definition (59) might fail to make sense if the domain of $\phi_{\mathcal{C}}^{-1}$ is not convex. The curve $\chi_{\mathcal{C}}$ can therefore not be defined for all charts of all space-times.

Finally, definition (59) makes clear that the curve $\chi_{\mathcal{C}}$ depends on $\mathcal{C}$ because the mapping $\phi_{\mathcal{C}}$ itself is part of the definition of chart $\mathcal{C}$.

We will now give a covariant definition of the curve $\xi_{\mathcal{C}}$. To that purpose, let us introduce the connection $\Gamma_{\mathcal{C}}$, the coefficients of which identically vanish in chart $\mathcal{C}$. This connection obviously depends on $\mathcal{C}$ because its coefficients in a different chart $\mathcal{C}'$ do not generically vanish, which proves that $\Gamma_{\mathcal{C}} \neq \Gamma_{\mathcal{C}'}$. Of particular importance is the following property of $\Gamma_{\mathcal{C}}$. If one parallel transports an arbitrary tensor with the connection $\Gamma_{\mathcal{C}}$ from a given point A to a given point B along a certain curve $\mathcal{L}$, the result will actually not depend on $\mathcal{L}$ because the Riemann curvature tensor of $\Gamma_{\mathcal{C}}$ vanishes identically. This can be most easily proven by a direct calculation using the chart $\mathcal{C}$.

Let now $\sigma_{M,U}$ be the collection of geodesics to be averaged, $N(\lambda,\omega)$ a running point on the geodesic $\gamma(\omega)$ and consider the family $\mathcal{F}$ of curves (or set of points)

$C_\lambda = \{N(\lambda, \omega), \omega \in \Omega\}$. This family depends on the collection $\sigma_{M,U}$, but not on the chart $\mathcal{C}$. Now, for each λ, choose an arbitrary point $Q(\lambda)$ (for example on C_λ) and, for each ω, parallel transport the vector $p(\lambda, \omega)$, cotangent to the geodesic $\gamma(\omega)$ at point $N(\lambda, \omega)$, from point $N(\lambda, \omega)$ to point $Q(\lambda)$ (for example, along the curve C_λ) using the connection $\Gamma_{\mathcal{C}}$. That way, you obtain for each ω a *bona fide* cotangent vector at point $Q(\lambda)$. And, by a preceding remark, the result does not depend on the curve along which the covector $p(\lambda, \omega)$ is parallel transported to point $Q(\lambda)$.

Compute then the average[7] $\tau(\lambda)$ of all the obtained vectors cotangent to the space-time at point $Q(\lambda)$. This is a possible (and intrinsic) operation because the space cotangent to the space-time manifold at point $Q(\lambda)$ is a vector space.

Let us now parallel transport this covector $\tau(\lambda)$ to every point of space-time using again the connection $\Gamma_{\mathcal{C}}$. The resulting covector field $\hat{\tau}(\lambda, .)$ is uniquely defined because, as already noted, the parallel transport of a tensor with $\Gamma_{\mathcal{C}}$ between points does not depend on the curve along which the transport is carried out. Note also that, for the same reason, the covector field $\hat{\tau}(\lambda, .)$ does not depend on the choice of $Q(\lambda)$ either.

The mean metric $\bar{g}$, which is intrinsic to the statistical ensemble Σ, can then be used to generate the tangent vector field u dual to $\hat{\tau}$. The curve $\xi_{\mathcal{C}}$ is then defined as the integral curve to u which passes through point M.

This definition makes clear that $\xi_{\mathcal{C}}$ is not an intrinsic object because the curve depends on $\mathcal{C}$ through the connection $\Gamma_{\mathcal{C}}$.

A final remark. It may be tempting to try defining an intrinsic average of the geodesics in $\sigma_{M,U}$ by simply replacing the connection $\Gamma_{\mathcal{C}}$ entering the definition of $\xi_{\mathcal{C}}$ by an intrinsically defined connection over the mean space-time, for example the connection $\bar{\Gamma}$ of the mean metric. But this connection has a generically non-vanishing curvature tensor and, in general, the parallel transport of a tensor between two points using $\bar{\Gamma}$ will depend on the curve along which the transport is carried out. When implemented with $\bar{\Gamma}$ instead of $\Gamma_{\mathcal{C}}$, the whole preceding procedure thus produces a result that depends a priori on both the choice of $Q(\lambda)$ and on the choice of the curves used in the various parallel transports. Even if one chooses these curves to be geodesics of the mean metric, the choice of point $Q(\lambda)$ remains rather arbitrary; therefore, the method does not seem to permit an intrinsic definition of the average of $\sigma_{M,U}$.

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[7] Over ω.